Introduction to mathematics for economists

Exercise Sheet #3

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During the tutorials, we will solve indefinite integrals 8 and 11, definite integrals 1 and 3, the second price-auction exercise, the waiting-time exercise, and the wage bargaining exercise. So you must prioritize these exercises. But you will be provided a full answer sheet.

Indefinite Integrals

Indefinite Integral

By **indefinite integral**, we mean an antiderivative (or primitive) of the function f. For instance, the antiderivative of $f(x) = x$ is:

$$\int x\,dx = \frac{x^2}{2} + C$$

where C is just a constant. We call it *indefinite* because we do not specify the bounds — we only look for the primitive.

Compute the following indefinite integrals (do not forget to specify the definition set).

1. $\displaystyle\int x^\alpha\,dx$, with $\alpha \in \mathbb{R}$.
2. $\displaystyle\int x^2 e^{-x}\,dx$.
3. $\displaystyle\int x\ln(x)\,dx$.
4. $\displaystyle\int \cos^2(x)\,dx$.
5. $\displaystyle\int \frac{x}{x^2+1}\,dx$.
6. $\displaystyle\int \frac{1}{x^2-1}\,dx$.
7. $\displaystyle\int (ax+b)^n\,dx$, with $a \neq 0$, $b \in \mathbb{R}$, $n > 0$.
8. $\displaystyle\int \frac{\ln(x)}{x}\,dx$.
9. $\displaystyle\int \frac{\ln^n(x)}{x}\,dx$, with $n \in \mathbb{N}$.
10. $\displaystyle\int \frac{2x+1}{x^2+x+1}\,dx$.
11. $\displaystyle\int \frac{1}{1+x^2}\,dx$.

Definite Integrals

Compute the following integrals, when they are well defined.

1. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx.$
2. $\displaystyle\int_1^\infty \frac{1}{x}\,dx.$
3. $\displaystyle\int_0^\infty e^{-x}\,dx.$
4. $\displaystyle\int_e^\infty \frac{1}{x(\ln x)^2}\,dx.$

Leibniz Rule and Applications

Let

$$F(t) = \int_t^{t^2} \frac{x^2}{2}\, t \, dx.$$

1. Compute $F'(t)$ **without** computing the integral first.
2. Compute the explicit value of the integral.
3. Check that the derivative of the explicit integral matches the result from Exercise 1.

Second-Price Auction

We consider an auction environment consisting of an auctioneer who wants to sell a good (the auctioneer does not value the object). There are 2 bidders each with i.i.d. private valuation $v_i \sim F$ with support $[\underline{v}, \overline{v}]$. Bidders simultaneously and privately submit bids $b_1, b_2 \in \mathbb{R}_+$. Assume that players submit bids using a symmetric strategy $b_i = b(v_i)$, which is strictly increasing, continuous and differentiable in one's valuation. The winning bidder pays the second-highest bid.

4. Write down bidder i's expected payoff $u_i(v_i, b_i, b_{-i})$.
5. Show that truthful bidding is optimal for bidder i.

Probability: Expectations and Variance

Continuous Random Variables

A **continuous random variable** X is described by a **probability density function** (PDF) $f : \mathbb{R} \to \mathbb{R}_+$ satisfying:

$$\int_{-\infty}^{+\infty} f(x)\, dx = 1.$$

The probability that X falls in an interval $[a, b]$ is:

$$\Pr(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

The **cumulative distribution function** (CDF) is defined as:

$$F(t) = \Pr(X \leq t) = \int_{-\infty}^{t} f(x)\, dx.$$

The **expectation** (mean) of X is:

$$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx.$$

The **variance** of X measures the spread around the mean:

$$\mathrm{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big] = \int_{-\infty}^{+\infty} (x - \mathbb{E}[X])^2\, f(x)\, dx = \mathbb{E}[X^2] - (\mathbb{E}[X])^2.$$

Waiting Times at a Service Desk

A firm operates a customer service desk. The waiting time T (in minutes) before a customer is served is modelled as a continuous random variable with probability density function:

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0,$$

where $\lambda > 0$ is a parameter capturing how fast the desk operates (a higher λ means shorter average wait). This is called the **exponential distribution** with parameter λ.

1. Verify that f is a valid PDF, i.e. show that $\displaystyle\int_0^{+\infty} \lambda e^{-\lambda t}\, dt = 1$.
2. Compute the CDF $F(t) = \Pr(T \leq t)$.
3. A customer is considered "well served" if they wait less than 3 minutes. A manager sets $\lambda = 1$. What is the probability that a customer waits more than 3 minutes?
4. Compute $\mathbb{E}[T]$ and interpret it in terms of the firm's service desk.
5. Compute $\mathrm{Var}(T)$ using the identity $\mathrm{Var}(T) = \mathbb{E}[T^2] - (\mathbb{E}[T])^2$. *(Hint: compute* $\mathbb{E}[T^2] = \int_0^{\infty} t^2 \lambda e^{-\lambda t}\, dt$ *by integration by parts.)*

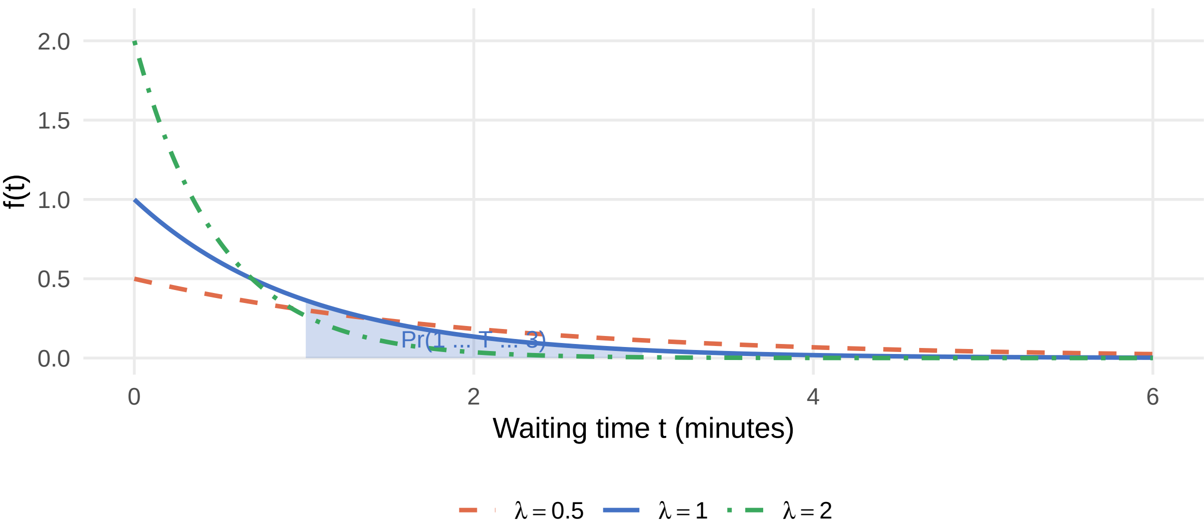

Figure 1

Economic Applications

Consumer and Producer Surplus

Consumer and Producer Surplus

Consider a market with inverse demand $P_D(q)$ and inverse supply $P_S(q)$, and let q^* be the equilibrium quantity at price p^*.

- The **consumer surplus** is the aggregate gain to buyers:

$$CS = \int_0^{q^*} (P_D(q) - p^*)\, dq.$$

- The **producer surplus** is the aggregate gain to sellers:

$$PS = \int_0^{q^*} (p^* - P_S(q))\, dq.$$

Consider a market for a consumption good with the following inverse demand and supply curves:

$$P_D(q) = 12 - q^2, \qquad P_S(q) = 2q + 2.$$

1. Find the equilibrium price p^* and quantity q^* (*hint: supply and demand must meet*).
2. Compute the consumer surplus CS.
3. Compute the producer surplus PS.
4. What is the total surplus $TS = CS + PS$? Briefly interpret this number.

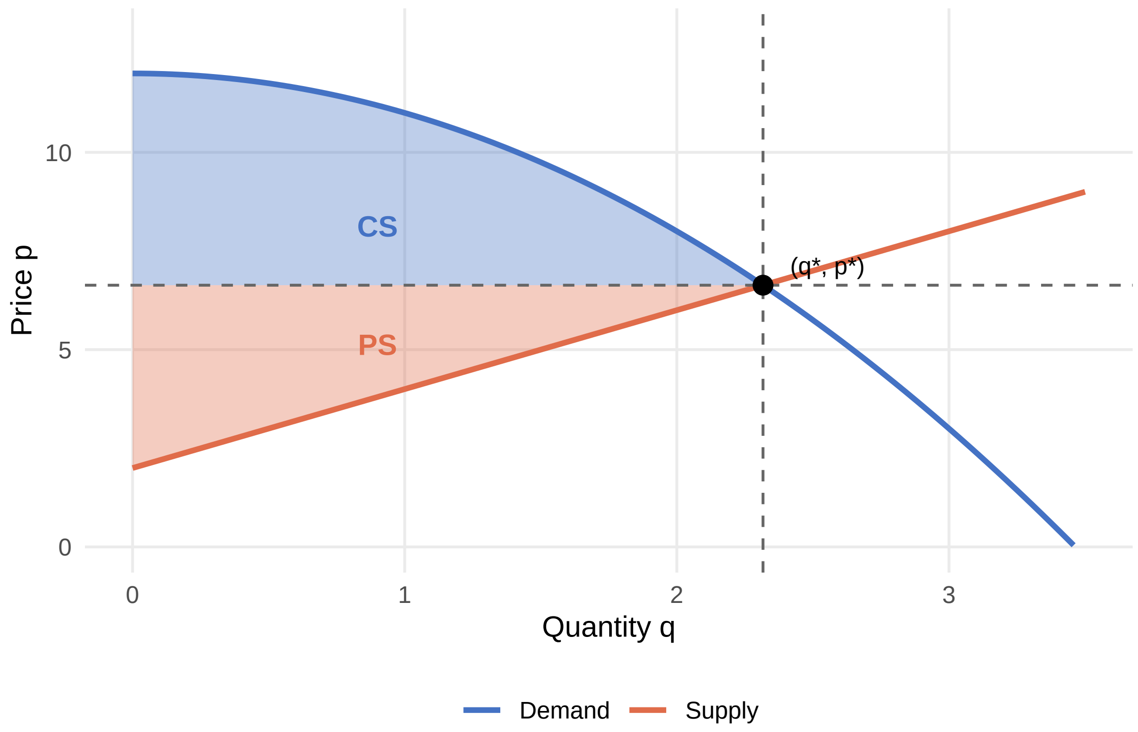

Figure 2

Lorenz Curve and Inequality

Lorenz Curve and Gini Coefficient

Let F be the CDF of income in a population, with mean μ. The **Lorenz curve** $L : [0,1] \to [0,1]$ maps the cumulative population share p to the corresponding cumulative income share:

$$L(p) = \frac{1}{\mu} \int_0^{F^{-1}(p)} x\, f(x)\, dx.$$

It satisfies $L(0) = 0$, $L(1) = 1$, and $L(p) \leq p$ for all p (the poorest p fraction always earns less than p fraction of total income). The **Gini coefficient** measures the area between the line of perfect equality and the Lorenz curve:

$$G = 1 - 2 \int_0^1 L(p)\, dp.$$

$G = 0$ means perfect equality; $G = 1$ means one person earns everything.

Suppose income y in a country follows a **power-law distribution** on $[1, e]$ with PDF:

$$f(y) = \frac{\alpha\, y^{\alpha - 1}}{e^{\alpha} - 1}, \quad y \in [1, e],$$

where $\alpha > 0$ controls the shape of inequality. Note that $y = 1$ is the minimum income and $y = e$ is the maximum.

1. Verify that f is a valid PDF on $[1, e]$.
2. Compute the mean income $\mu = \mathbb{E}[Y]$.
3. Compute the quantile function $F^{-1}(p)$, i.e. the income level below which a fraction p of the population falls. *(Hint: first compute the CDF $F(y)$, then invert it.)*

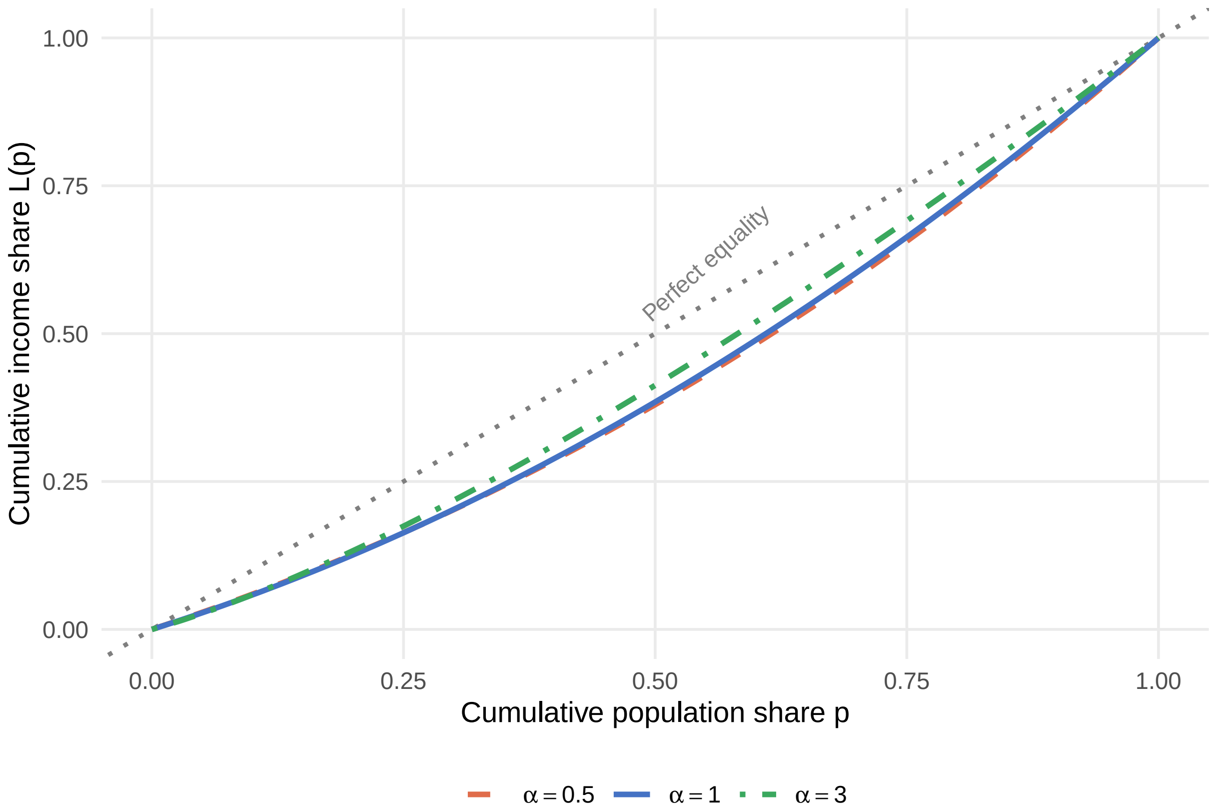

Figure 3

4. Show that the Lorenz curve is given by:

$$L(p) = \frac{\alpha}{\alpha+1} \cdot \frac{(p\,(e^{\alpha}-1)+1)^{(\alpha+1)/\alpha} - 1}{e^{\alpha+1}-1}.$$

5. What happens to the Lorenz curve as $\alpha \to \infty$? What does this imply economically? *(You may use the figure above to guide your intuition.)*

6. Compute the Gini coefficient $G = 1 - 2\int_0^1 L(p)\,dp$ for $\alpha = 1$.

Wage Bargaining

A firm and a worker negotiate over a wage w. The job generates a surplus S (the total value created if they reach an agreement), but this surplus is **uncertain**: it is drawn from a uniform distribution on $[0, \bar{s}]$, so its PDF is $f(s) = 1/\bar{s}$.

The worker has an outside option $\bar{u} > 0$ (e.g. unemployment benefits): they will only accept the job if $w \geq \bar{u}$. The firm will only offer the job if $S - w \geq 0$. Hence a deal is struck **only when** $S \geq \bar{u}$.

When a deal is struck, the parties solve the **Nash bargaining problem**: they choose w to maximise the Nash product

$$N(w, S) = (w - \bar{u})^{\beta} \cdot (S - w)^{1-\beta},$$

where $\beta \in (0, 1)$ is the worker's bargaining power.

1. Taking S as given (i.e. treating it as a known constant), find the wage $w^*(S)$ that maximises the Nash product $N(w, S)$. *(Hint: take the log of N, then differentiate with respect to w.)*

Figure 4

2. Show that the expected negotiated wage, conditional on a deal being struck, is:

$$\mathbb{E}[w^* \mid S \geq \bar{u}] = \frac{\int_{\bar{u}}^{\bar{s}} w^*(s)\, f(s)\, ds}{\Pr(S \geq \bar{u})}.$$

   Compute this explicitly.

3. How does the expected wage change with β? With $\bar{u}$? Provide economic intuition.
4. Compute the expected **firm profit** conditional on a deal, i.e. $\mathbb{E}[S - w^* \mid S \geq \bar{u}]$.